ADDITION THEOREMS

from the

STANDPOINT OF ABEL'S THEOREM.

submitted by

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INTRODUCTION

In this paper there is shown in detail the proofs of certain addition theorems from the standpoint of Abel's Theorem.

ABELIAN INTEGRALS

Any integral $I = \int R(xy)dx$ where $R(xy)$ is a rational function of x and y, where y is an algebraic function defined by an equation $F(xy) = 0$ is called an Abelian Integral attached to that curve. To obtain the determination of this integral it is necessary to assign a lower limit x_0 and a corresponding value y_0 is chosen among roots of equation $F(x_0 y) = 0$. On going from point x_0 to any point x by all possible paths, all values of the integral I are included in one of the formulas $I = I_K + m_1 w_1 + m_2 w_2 + \text{-------} m_r w_r$ (K = 1,2,3, ----- n) where I_1, I_2, ------- I_n are values of the integral which correspond to certain definite paths; m_1, m_2, ------ m_r, and w_1, w_2, ------- w_r, are periods. These periods are of two kinds; one kind results from loops described about the poles of $F(xy) = 0$; these are Polar Periods. The other comes from closed paths surrounding several critical points, called Cycles; these are called Cyclic Periods. The number of distinct Cyclic periods depends only on the algebraic function considered $F(xy) = 0$; it is equal to 2p, where p denotes the deficiency of the curve. From point of view of singularities 3 classes of Abelian Integrals are distinguished.

(1) Those which remain <u>finite</u> in the neighborhood of every value of x are called <u>First Kind</u>.

(2) Those which have a <u>single pole</u> are called the <u>Second Kind</u>.

(3) Those having <u>two logarithmic singular</u> points are called <u>Third Kind</u>.

ABEL'S THEOREM

Consider a plane curve $F(xy) = 0$ and let $\phi(xy) = 0$ be the equation of another plane algebraic curve. These curves have n points in common $(x_1 y_1)$, $(x_2 y_2)$ ----- the number n equaling the product of degrees of the two curves. Let $R(xy)$ be a rational function and consider the following sum

(1) $I = \sum_{i=1}^{n} \int_{x_0 y_0}^{x_i y_i} R(xy)dx$ where $\int_{x_0 y_0}^{x_i y_i} R(xy)dx$ denotes the Abelian integral taken from a fixed point x_0 to a point x_i along a path which leads y from an initial value y_0 to final of y_i, the initial value y_0 being same for all these integrals.

Suppose, now, some of the coefficients a, b, c ------of $\phi(xy) = 0$ are variable. When these coefficients vary continuously the points x_i vary continuously, and if none of these pass thru a discontinuity of $\int R(xy)dx$, the sum I itself varies continuously, provided we follow the continuous variation of each of the integrals contained in it along the entire path descrited by corresponding upper limit. The sum I is therefore a function of the <u>parameters</u> a, b, c, -------, whose analytic form is considered next.

Denoting the total differential of any function U by dU, with respect to variables a, b, c, -----------, we have,

$dU = \frac{\partial U}{\partial a} da + \frac{\partial U}{\partial b} db +$ ---- . By (1) $dI = \sum_{i=1}^{n} R(x_i y_i) dx_i$ and from the two relations $F(x_i y_i) = 0$ and $\phi(x_i y_i) = 0$ there is derived

$$\frac{\partial F}{\partial x_i} dx_i + \frac{\partial F}{\partial y_i} dy_i = 0$$

and

$$\frac{\partial \phi}{\partial x_i} dx_i + \frac{\partial \phi}{\partial y_i} dy_i + d\phi_i = 0$$

and, consequently $dx_i = \psi(x_i y_i) d\phi_i$ where $\psi(x_i y_i)$ is a rational function of x_i, y_i, a, b, c, -------- and where ϕ_i stands for $\phi(x_i y_i)$, we obtain,

$$dI = \sum_{i=1}^{n} R(x_i y_i)\, \psi(x_i y_i)\, d\phi_i$$

The coefficient of da is a rational symmetric function of the coordinates of n points $(x_i y_i)$ common to the two curves $F(xy) = 0$ and $\phi(xy) = 0$. The theory of elimination tells us that this function is a rational function of coefficients of the two polynomials $F(xy) = 0$ and $\phi(xy) = 0$ and consequently a rational function of a, b, c, --------- Evidently the same is true of the coefficients of db, dc, etc. and I will be obtained by integration of a total differential

$I = \int (f_1\, da + f_2\, db + f_3\, dc)$ where f_1, f_2, f_3, ------- are rational functions of a, b, c, ---------- and the integration cannot introduce any other transcendentals other than logarithms.

The sum I is therefore equal to a Rational Function of the Coefficients a, b, c, ------ plus a sum of logarithms of rational functions of the coefficients a, b, c, ------, each of these logarithms being multiplied by a constant factor.

The above is a statement of Abel's Theorem in its most general form.

In geometric language this can be stated as follows: The sum of the values of an Abelian Integral, taken from a common origin to the n points of intersection of the given curve with a variable curve of degree m, $\phi(xy) = 0$, is equal to a rational function of the coefficients of $\phi(xy) = 0$ plus a sum of a finite number of logarithms of rational functions of the same coefficients, each logarithm being multiplied by a constant factor.

Abel's Theorem has a definite meaning only if paths described by x_1, x_2, x_3, ------- x_n are taken into account.

The theorem becomes simpler when the Abelian integral is of the first kind. If f_1, f_2, f_3, -------f_k were not identically zero, it would be possible to find a system of values $a = a'$, $b = b'$, $c = c'$, -------- for which I would become infinite. Let (x_1', y_1') ----- (x_n', y_n') be the intersections of the curves F(xy) and $\phi(xy)$ which correspond to the parametric values a', b', c'.

Then $\int_{x_0 y_0}^{xy} R(xy)dx$ becomes infinite when the upper limit approaches one of the points (x_i', y_i'); Since this is impossible when the Integral is of the first kind, we have $dI = 0$, and, consequently I = a constant, as a, b, c, vary continuously.

Abel's Theorem can then be stated as follows: Given a fixed curve $F(xy) = 0$ and a variable curve $\phi(xy) = 0$ of degree m, the sum of the increments of an Abelian Integral of the first kind attached to the curve $F(xy)$ along the continious curves described by the points of intersection of $F(xy) = 0$ with $\phi(xy) = 0$ is equal to zero.

Note: References for theory are made especially to the following three texts which were consulted:

Goursat's: Mathematical Analysis Vol.2 Part1

H.F. Baker's: Abelian Functions.

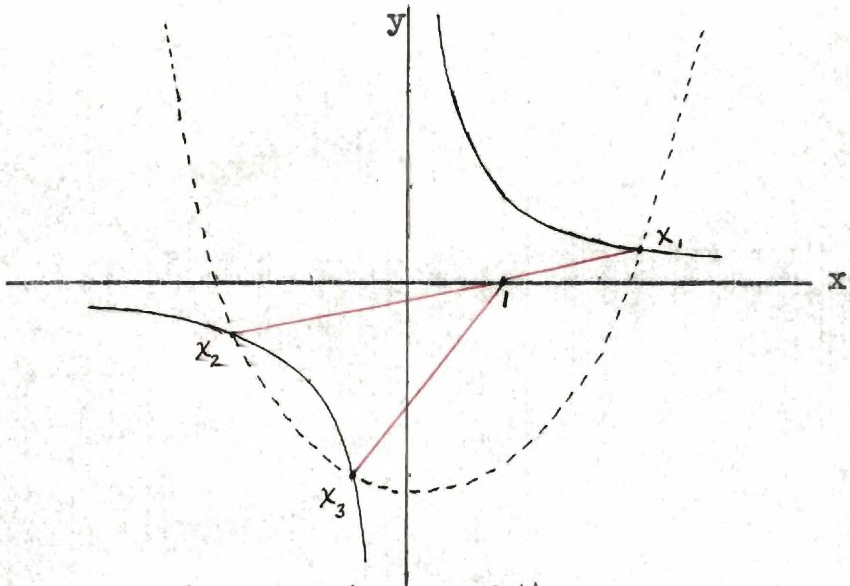
Harkness and Morley: Theory of Functions.

SECTION (1)

THE ADDITION THEOREM FOR $\log x = \int_1^x \frac{dx}{x}$

(1) Let $y = \frac{1}{x}$ represent the fixed curve

(2) And $y = x^2 + ax + b$ the variable curve



$$U(ab) = \int_1^{x_1(ab)} \frac{dx}{x} + \int_1^{x_2(ab)} \frac{dx}{x} + \int_1^{x_3(ab)} \frac{dx}{x}$$ by Abel's theorem

(3) $U_a = y_1 x_{1a} + y_2 x_{2a} + y_3 x_{3a}$

On eliminating y from (1) and (2) we have,

(4) $\phi(abx) = x^3 + ax^2 + bx - 1$ whose roots are the abscissas of the intersections of (1) and (2)

Now $\phi'_x \cdot x_a + \phi_a = \phi'_x \cdot x_a + x^2 = 0$

Therefore $x_a = -\frac{x^2}{\phi'_x}$

Substitution in equation (3) gives

(5) $U_a = \frac{1}{x_1} \cdot \frac{-x_1^2}{\phi'_{x_1}} + \frac{1}{x_2} \cdot \frac{-x_2^2}{\phi'_{x_2}} + \frac{1}{x_3} \cdot \frac{-x_3^2}{\phi'_{x_3}}$

Or $U_a = -\left[\frac{x_1}{\phi'_{x_1}} + \frac{x_2}{\phi'_{x_2}} + \frac{x_3}{\phi'_{x_3}}\right]$

where, $\phi'_{x_1} = 3x_1^2 + 2ax_1 + b$

$\phi'_{x_2} = 3x_2^2 + 2ax_2 + b$

$\phi'_{x_3} = 3x_3^2 + 2ax_3 + b$

Therefore $U_a = -\left[\frac{x_1}{3x_1^2+2ax_1+b} + \frac{x_2}{3x_2^2+2ax_2+b} + \frac{x_3}{3x_3^2+2ax_3+b}\right]$

(6) By elementary operations we have as the numerator of U_a,

$- [9x_1 x_2 x_3 (x_2x_3 + x_1x_3 + x_1x_2) + 6ax_1x_2x_3(x_1 + x_2 + x_3) + 3bx_1 ($
$x_3^2 + x_1x_2 + x_1x_3) + 6ax_1x_2x_3(x_2 + x_3 + x_2) + 4a^2x_1x_2x_3(1+1+1) +$
$2abx_1(x_3 + x_2 + x_3) + 3bx_2(x_1x_2 + x_3^2 + x_2x_3) + 2abx_2(x_1 + x_3 + x_3)$
$+ b^2(x_1 + x_2 + x_3)]$

We now simplify this expression as follows:

From (4) we have,

(7) $$\left\{\begin{array}{l} x_1 + x_2 + x_3 = -a \\ x_1x_2 + x_1x_3 + x_2x_3 = b \\ x_1x_2x_3 = 1 \end{array}\right\}$$

Substituting relations from (7) in (6) we get,

$-[9b - 6a^2 + 3bx_1(x_3^2 + x_1x_2 + x_1x_3) - 6a^2 + 12a^2 + 4ab^2 + 3bx_2(x_1x_2 +$
$x_3^2 + x_2x_3) - ab^2] =$

(8) $-[9b + 3ab^2 + 3b(x_1x_3^2 + x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_2x_3^2 + x_2^2x_3)]$

From (7) $(x_1x_2 + x_1x_3 + x_2x_3)(x_1 + x_2 + x_3) = x_1^2x_2 + x_1^2x_3 +$
$x_1x_2^2 + x_2^2x_3 + x_1x_3^2 + x_2x_3^2 + 3x_1x_2x_3$

Therefore $(b)(-a) = x_1x_3^2 + x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_2x_3^2 +$
$x_2^2x_3 + 3$ <u>or</u> $x_1x_3^2 + x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_2x_3^2 + x_2^2x_3 = -ab-3$

Substitution in (8) gives for the numerator of U_a,
$-[9b + 3ab^2 + 3b(-ab - 3)] = -[9b + 3ab^2 - 3ab^2 - 9b] = 0$

Hence $U_a = 0$

Likewise, for U_b, we have,

(9) $U_b = y_1x_{1b} + y_2x_{2b} + y_3x_{3b}$

and $\phi'_x \cdot x_b + \phi_b = \phi'_x \cdot x_b + x = 0$

Therefore $x_b = -\frac{x}{\phi'_x}$

Substitution in (9) gives,

$$U_b = \frac{1}{x_1}\cdot\frac{-x_1}{\phi'_{x_1}} + \frac{1}{x_2}\cdot\frac{-x_2}{\phi'_{x_2}} + \frac{1}{x_3}\cdot\frac{-x_3}{\phi'_{x_3}}$$

Or $$U_b = -\left[\frac{1}{\phi'_{x_1}} + \frac{1}{\phi'_{x_2}} + \frac{1}{\phi'_{x_3}}\right]$$

(10) $$U_b = -\left[\frac{1}{3x_1^2+2ax_1+b} + \frac{1}{3x_2^2+2ax_2+b} + \frac{1}{3x_3^2+2ax_3+b}\right]$$

By elementary reductions the numerator of (10) becomes,

(11) $$-[9(x_1^2x_2^2+x_1^2x_3^2+x_2^2x_3^2) + 6a(x_2x_3^2+x_1x_3^2+x_3x_1^2+x_2x_1^2+x_3x_2^2+x_1x_2^2) + 6b(x_1^2+x_2^2+x_3^2) + 4a^2(x_2x_3+x_1x_3+x_1x_2) + 4ab(x_1+x_2+x_3) + 3b^2]$$

Squaring the first equation of (7) and transposing, we obtain: $$x_1^2 + x_2^2 + x_3^2 = a^2 - 2b$$

And from U_a it has been shown that,

$$x_1x_3^2 + x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_2x_3^2 + x_2^2x_3 = -ab - 3$$

Also, from the second equation in (7)

$$x_1^2x_2^2 + x_2^2x_3^2 + x_1^2x_3^2 = b^2 - 2x_1x_2x_3(x_1+x_2+x_3)$$

Therefore $$x_1^2x_2^2 + x_2^2x_3^2 + x_1^2x_3^2 = b^2 + 2a$$

Hence, substituting relations from cubic in (11) we obtain; for the numerator of U_b,

$$-[9(b^2+2a) + 6a(-ab-3) + 6b(a^2-2b) + 4a^2(b) + 4ab(-a) + 3b^2] = -[9b^2 + 18a - 6a^2b - 18a + 6a^2b - 12b^2 + 4a^2b - 4a^2b + 3b^2] = 0$$

Therefore $U_b = 0$

Hence, $U_a = 0$, $U_b = 0$, and consequently

$$U(ab) = \text{a constant} = C$$

And $$\int_1^{x_1}\frac{dx}{x} + \int_1^{x_2}\frac{dx}{x} + \int_1^{x_3}\frac{dx}{x} = C$$

As no restrictions were placed on a and b we take

$a = -3$, $b = 3$ The intersection then becomes,

$x^3 - 3x^2 + 3x - 1 = 0$ with a triple root $x_1 = x_2 = x_3 = 1$

Therefore we have, $\int_1^1 \frac{dx}{x} + \int_1^1 \frac{dx}{x} + \int_1^1 \frac{dx}{x} = C$

Hence $0 = C$

And $$\int_1^{x_1} \frac{dx}{x} + \int_1^{x_2} \frac{dx}{x} + \int_1^{x_3} \frac{dx}{x} = 0$$

(12) Transposing, $\int_1^{x_1} \frac{dx}{x} + \int_1^{x_2} \frac{dx}{x} = -\int_1^{x_3} \frac{dx}{x} = \int_1^{\frac{1}{x_3}} \frac{dx}{x}$

As $x_1 x_2 x_3 = 1$ we get $x_3 = \frac{1}{x_1 x_2}$, therefore $\frac{1}{x_3} = x_1 x_2$

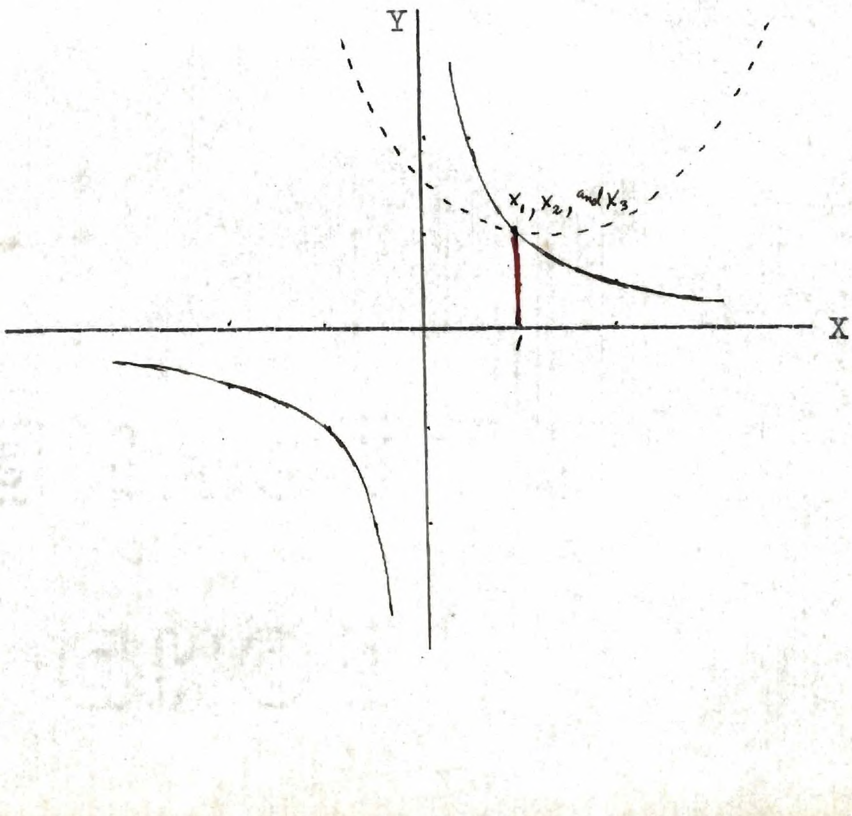
Hence, substituting in (12) we get,

$$\int_1^{x_1} \frac{dx}{x} + \int_1^{x_2} \frac{dx}{x} = \int_1^{x_1 x_2} \frac{dx}{x}$$

Or, $\log x_1 + \log x_2 = \log x_1 x_2$, which is

THE ADDITION THEOREM FOR LOGARITHMS

The curves below show how the intersection occurs for the special case chosen, $a = -3$, $b = 3$

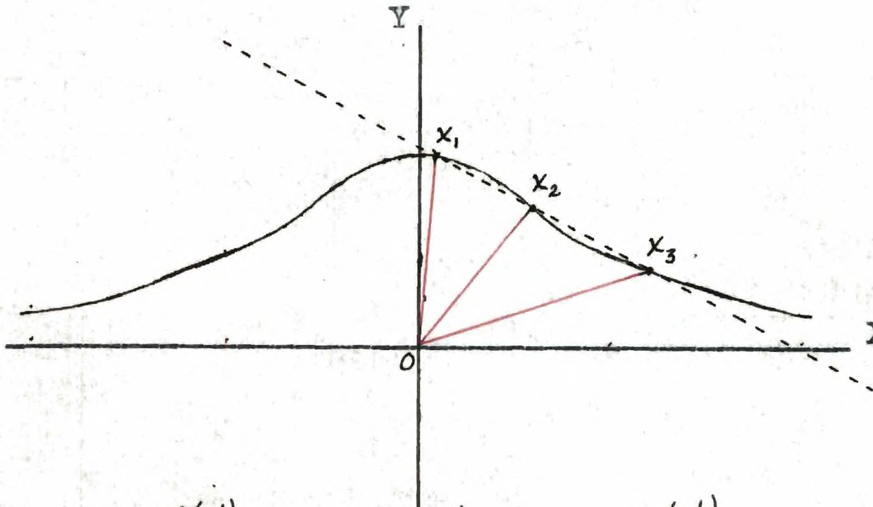


SECTION (2)

THE ADDITION THEOREM FOR ARC TANGENT $x = \int_0^x \frac{dx}{x^2+1}$

(1) Let $y = \frac{1}{x^2+1}$ represent the fixed curve

(2) And $y = ax + b$ the variable curve



$$U(ab) = \int_0^{x_1(ab)} \frac{dx}{x^2+1} + \int_0^{x_2(ab)} \frac{dx}{x^2+1} + \int_0^{x_3(ab)} \frac{dx}{x^2+1}$$

On eliminating y between (1) and (2) we get,

(3) $ax^3 + bx^2 + ax + b - 1 = 0$ whose roots are the abscissas of the intersections of (1) and (2)

Proceeding as in Section (1), we have

(4) $U_a = y_1 x_{1a} + y_2 x_{2a} + y_3 x_{3a}$

Since $\phi(abx) = ax^3 + bx^2 + ax + b - 1$

Therefore $\phi'_x = 3ax^2 + 2bx + a$

Here $\phi'_x \cdot x_a + \phi_a = \phi'_x \cdot x_a + (x^3 + x) = 0$

Hence $x_a = -\frac{x^3 + x}{\phi'_x} = -\frac{x(x^2+1)}{\phi'_x} = -\frac{x}{y\,\phi'_x}$

Substitution in (4) gives,

$$U_a = y_1 \cdot \frac{-x_1}{y_1 \phi'_{x_1}} + y_2 \cdot \frac{-x_2}{y_2 \phi'_{x_2}} + y_3 \cdot \frac{-x_3}{y_3 \phi'_{x_3}}$$

$$U_a = -\left[\frac{x_1}{3ax_1^2 + 2bx_1 + a} + \frac{x_2}{3ax_2^2 + 2bx_2 + a} + \frac{x_3}{3ax_3^2 + 2bx_3 + a}\right]$$

Hence, we get, as the numerator of U_a ,

(5) $-[x_1(3ax_2^2 + 2bx_2 + a)(3ax_3^2 + 2bx_3 + a) + x_2(3ax_1^2 + 2bx_1 + a)$
$(3ax_3^2 + 2bx_3 + a) + x_3(3ax_2^2 + 2bx_2 + a)(3ax_1^2 + 2bx_1 + a)\ ]$

By elementary operations (5) can be written:

(6) $-\ [9a^2(x_1x_2x_3)(x_1x_2 + x_1x_3 + x_2x_3) + 6ab(x_1x_2x_3)(x_2 + x_3 + x_2 + x_1 + x_1 + x_3) + 4b^2(x_1x_2x_3 + x_1x_2x_3 + x_1x_2x_3) + 2ab(x_2x_3 + x_2x_3 + x_1x_2 + x_1x_3 + x_1x_2 + x_1x_3) + a^2(x_3 + x_2 + x_1) + 3a^2(x_2^2x_3 + x_2x_3^2 + x_1x_2^2 + x_1^2x_3 + x_1^2x_2 + x_1x_3^2)\ ]$

From (3) we have,

$ax^3 + bx^2 + ax + b - 1 = 0$, or $x^3 + \frac{b}{a}x^2 + x + \frac{b-1}{a} = 0$

Therefore, we obtain,

$$(7) \qquad \left\{\begin{aligned} x_1 + x_2 + x_3 &= -\frac{b}{a} \\ x_1x_2 + x_2x_3 + x_1x_3 &= 1 \\ x_1x_2x_3 &= \frac{1-b}{a} \end{aligned}\right\}$$

From (7) by multiplying first equation by the second we get, $x_2^2x_3 + x_2x_3^2 + x_1x_2^2 + x_1^2x_3 + x_1^2x_2 + x_1x_3^2 = (x_1 + x_2 + x_3)(x_1x_2 + x_1x_3 + x_2x_3) - 3x_1x_2x_3 = -\frac{b}{a} - 3\left(\frac{1-b}{a}\right) = \frac{-b-3+3b}{a} = \frac{-3+2b}{a}$

Substituting these relations in (6) we get, as the numerator of U_a ,

$-\ [\ 9a(1-b) + 6b(1-b)(-\frac{2b}{a}) + 12b^2(\frac{1-b}{a}) + 4ab - ab - 9a + 6ab\ ] = -\ [\ 9a - 9ab - \frac{12b^2}{a}(1-b) + \frac{12b^2}{a}(1-b) - 9a + 9ab] = 0$

Therefore $U_a = 0$

To determine U_b we have,

(8) $U_b = y_1x_{1b} + y_2x_{2b} + y_3x_{3b}$

$$\phi\,(abx) = ax^3 + bx^2 + ax + b - 1$$

Therefore $\phi'_x = 3ax^2 + 2bx + a$, whence

$$\phi'_x \cdot x_b + \phi_b = \phi'_x \cdot x_b + (x^2 + 1) = 0$$

Therefore $x_b = -\dfrac{x^2+1}{\phi'_x} = -\dfrac{1}{y\,\phi'_x}$

Substitution in (8) gives,

$$U_b = y_1 \cdot \frac{-1}{y_1 \phi'_{x_1}} + y_2 \cdot \frac{-1}{y_2 \phi'_{x_2}} + y_3 \cdot \frac{-1}{y_3 \phi'_{x_3}} =$$

$$(9) \qquad -\left[\frac{1}{3ax_1^2 + 2bx_1 + a} + \frac{1}{3ax_2^2 + 2bx_2 + a} + \frac{1}{3ax_3^2 + 2bx_3 + a}\right]$$

Clearing (9) the numerator of U_b becomes,

$$(10) \quad -\,[(3ax_2^2 + 2bx_2 + a)(3ax_3^2 + 2bx_3 + a) + (3ax_1^2 + 2bx_1 + a)(3ax_3^2 + 2bx_3 + a) + (3ax_1^2 + 2bx_1 + a)(3ax_2^2 + 2bx_2 + a)]$$

By elementary reductions (10) can be written:

$$(11) \quad -\,[9a^2(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2) + 6ab(x_1x_2^2 + x_1x_3^2 + x_2^2x_3 + x_1^2x_2 + x_1^2x_3 + x_2x_3^2) + 3a^2(x_2^2 + x_3^2 + x_2^2 + x_1^2 + x_1^2 + x_3^2) + 4b^2(x_1x_2 + x_1x_3 + x_2x_3) + 2ab(x_2 + x_3 + x_2 + x_1 + x_1 + x_3) + 3a^2]$$

From (7) we get,

$$x_1^2 + x_2^2 + x_3^2 = \frac{b^2}{a^2} - 2(x_1x_2 + x_1x_3 + x_2x_3) = \frac{b^2}{a^2} - 2 = \frac{b^2 - 2a^2}{a^2}$$

$$x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2 = 1 - 2(x_1x_2x_3)(x_1 + x_2 + x_3) = \frac{a^2 + 2b - 2b^2}{a^2}$$

Substituting these relations, and also the ones under (7) in (11) we get, for the numerator of U_b ,

$$-\left[9a^2\left(\frac{a^2 + 2b - 2b^2}{a^2}\right) + 6ab\left(\frac{-3+2b}{a}\right) + 6a^2\left(\frac{b^2 - 2a^2}{a^2}\right) + 4b^2 + 4ab\left(-\frac{b}{a}\right) + 3a^2\right] = -\,[9a^2 + 18b + 12b^2 - 18b^2 - 18b + 6b^2 - 12a^2 + 4b^2 - 4b^2 + 3a^2] = 0$$

, and consequently $U_b = 0$

Hence $U_a = 0$, $U_b = 0$

And, consequently, $U(ab)$ = a constant = C

Then, we have (12) $$\int_0^{x_1}\frac{dx}{x^2+1}+\int_0^{x_2}\frac{dx}{x^2+1}+\int_0^{x_3}\frac{dx}{x^2+1}=C$$

If we let $a = 0$ and $b = 1$ in (3) it reduces to $x^2 = 0$, whose roots are: $x_1 = x_2 = 0$, $x_3 = \infty$

and (12) becomes $$\int_0^{0}\frac{dx}{x^2+1}+\int_0^{0}\frac{dx}{x^2+1}+\int_0^{\infty}\frac{dx}{x^2+1}=C$$

Since the first two integrals vanish, we get

$$\int_0^{\infty}\frac{dx}{x^2+1}=C=\frac{\pi}{2}.$$ Substituting in (12) we obtain:

$$\int_0^{x_1}\frac{dx}{x^2+1}+\int_0^{x_2}\frac{dx}{x^2+1}+\int_0^{x_3}\frac{dx}{x^2+1}=\frac{\pi}{2}$$

Therefore $$\int_0^{x_1}\frac{dx}{x^2+1}+\int_0^{x_2}\frac{dx}{x^2+1}=\frac{\pi}{2}-\int_0^{x_3}\frac{dx}{x^2+1}$$

Since $\frac{\pi}{2}-\operatorname{Tan}^{-1}x_3=\operatorname{Tan}^{-1}\frac{1}{x_3}$, hence we have

$$\int_0^{x_1}\frac{dx}{x^2+1}+\int_0^{x_2}\frac{dx}{x^2+1}=\int_0^{\frac{1}{x_3}}\frac{dx}{x^2+1} \tag{13}$$

From the second equation under (7) we have,

$x_1x_2+x_1x_3+x_2x_3=1$. From this $x_1x_2+(x_1+x_2)x_3=1$, and

$x_3=\frac{1-x_1x_2}{x_1+x_2}$. Therefore $\frac{1}{x_3}=\frac{x_1+x_2}{1-x_1x_2}$

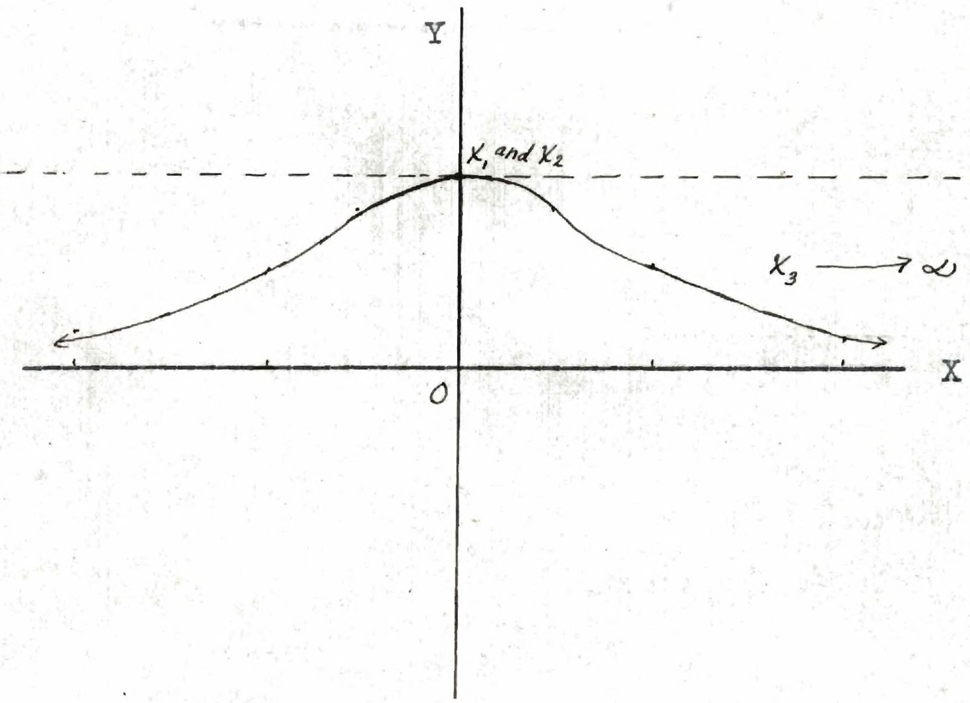
Hence (13) becomes,

$$\int_0^{x_1}\frac{dx}{x^2+1}+\int_0^{x_2}\frac{dx}{x^2+1}=\int_0^{\frac{x_1+x_2}{1-x_1x_2}}\frac{dx}{x^2+1}$$

Or, ARC TANGENT x_1 + ARC TANGENT x_2 = ARC TANGENT $\frac{x_1+x_2}{1-x_1x_2}$

Which is, "THE ADDITION THEOREM FOR THE ARC TANGENT".

The values taken for a and b for determining the constant C, a = 0, b = 1, threw the straight line, used as a variable curve, in the position as shown on the graph below.

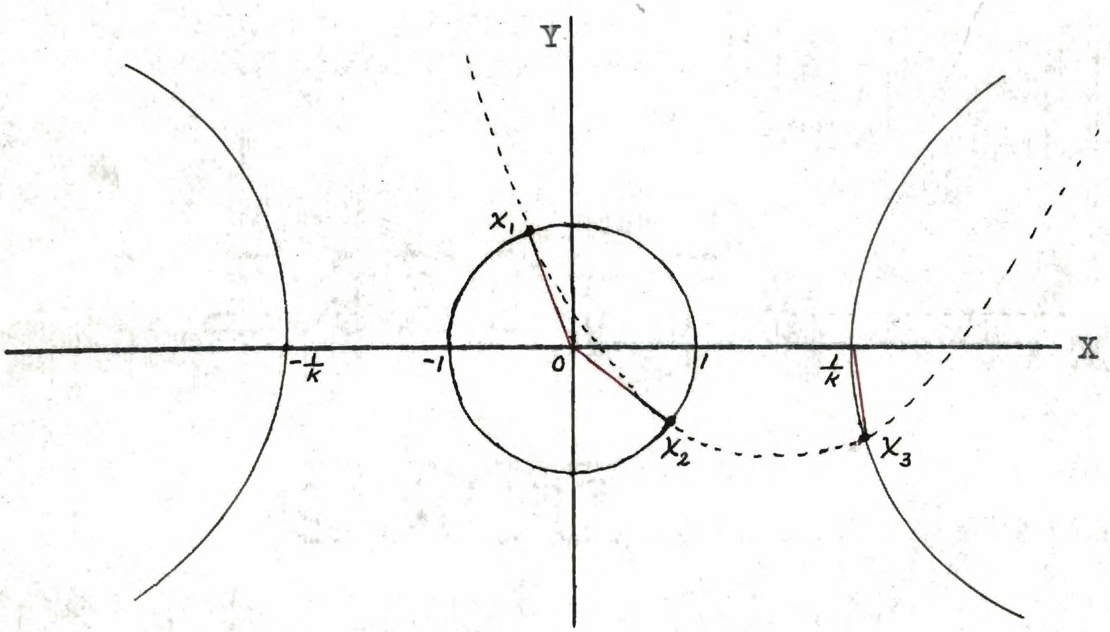


SECTION (3)

THE ADDITION THEOREM FOR LEGENDRE'S ELLIPTIC INTEGRAL OF THE FIRST KIND, $F(k,x) = \int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$

(1) Let $y = \sqrt{(1-x^2)(1-k^2x^2)}$ represent the fixed curve.

(2) And $y = kx^2 + ax + b$ the variable curve.



$$U(ab) = \int_0^{x_1(ab)} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} + \int_0^{x_2(ab)} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} + \int_{\frac{1}{k}}^{x_3(ab)} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

From (1) and (2) we obtain the cubic,

$2akx^3 + (a^2 + 2bk + k^2 + 1)x^2 + 2abx + b^2 - 1 = 0$ whose roots are the abscissas of the intersections of (1) and (2).

Therefore we have,

(3) $\phi(abx) = 2akx^3 + (a^2 + k^2 + 2bk + 1)x^2 + 2abx + b^2 - 1 = 0$

Proceeding as in SECTIONS (1) and (2), we have,

(4) $\quad U_a = y_1 x_{1a} + y_2 x_{2a} + y_3 x_{3a}$

Determining x_a, we have,

$$\phi'_x \cdot x_a + \phi_a = \phi'_x \cdot x_a + (2kx^3 + 2ax^2 + 2bx) = 0$$

Therefore $x_a = - \dfrac{2x(kx^2 + ax + b)}{\phi'_x}$

Where $\phi'_x = 6akx^2 + (a^2 + k^2 + 2bk + 1)2x + 2ab$

Hence, $x_a = - \dfrac{x(kx^2+ax+b)}{3akx^2+(a^2+k^2+2bk+1)x+ab} = \dfrac{-xy}{3akx^2+(a^2+k^2+2bk+1)x+ab}$

Substituting in (4) we obtain,

$$(5) \quad U_a = - \left[\frac{1}{y_1} \cdot \frac{x_1 y_1}{3akx_1^2+(a^2+k^2+2bk+1)x_1+ab} + \frac{1}{y_2} \cdot \frac{x_2 y_2}{3akx_2^2+(a^2+k^2+2bk+1)x_2+ab} \right.$$

$$\left. + \frac{1}{y_3} \cdot \frac{x_3 y_3}{3akx_3^2+(a^2+k^2+2bk+1)x_3+ab} \right]$$

consequently (5) becomes,

$$(6) \quad U_a = - \left[\frac{x_1}{3akx_1^2+(a^2+k^2+2bk+1)x_1+ab} + \frac{x_2}{3akx_2^2+(a^2+k^2+2bk+1)x_2+ab} \right.$$

$$\left. + \frac{x_3}{3akx_3^2+(a^2+k^2+2bk+1)x_3+ab} \right]$$

By elementary reductions the numerator of U_a becomes,

(7)

$$x_1 [9a^2k^2x_2^2x_3^2 + 3ak(a^2+k^2+2bk+1)(x_2^2x_3 + x_3^2x_2) + 3a^2bk(x_2^2 + x_3^2)$$
$$+ (a^2+k^2+2bk+1)^2x_2x_3 + a^2b^2 + ab(a^2+k^2+2bk+1)(x_2 + x_3)] +$$
$$x_2 [9a^2k^2x_1^2x_3^2 + 3ak(a^2+k^2+2bk+1)(x_1^2x_3 + x_3^2x_1) + 3a^2bk(x_1^2 + x_3^2) +$$
$$(a^2+k^2+2bk+1)^2x_1x_3 + a^2b^2 + ab(a^2+k^2+2bk+1)(x_1 + x_3)] +$$
$$x_3 [9a^2k^2x_1^2x_2^2 + 3ak(a^2+k^2+2bk+1)(x_1^2x_2 + x_2^2x_1) + 3a^2bk(x_1^2 + x_2^2) +$$
$$(a^2+k^2+2bk+1)^2x_1x_2 + a^2b^2 + ab(a^2+k^2+2bk+1)(x_1 + x_2)]$$

Which reduces to,

$$(8) \quad 9a^2k^2(x_1x_2x_3)(x_1x_3 + x_1x_2 + x_2x_3) + 6ak(a^2+k^2+2bk+1)(x_1x_2x_3)(x_1 + x_2 + x_3) + 3a^2bk(x_1x_2^2 + x_1x_3^2 + x_2x_1^2 + x_2x_3^2 + x_3x_1^2 + x_3x_2^2) + (a^2+k^2+2bk+1)^2(3)(x_1x_2x_3) + a^2b^2(x_1 + x_2 + x_3) + 2ab(a^2+k^2+2bk+1)(x_1x_2 + x_1x_3 + x_2x_3)$$

From (3) we obtain,

$$(9)\left\{\begin{array}{ll}
(a) & x_1 + x_2 + x_3 = -\dfrac{a^2+k^2+2bk+1}{2ak} \\[2ex]
(b) & x_1x_2 + x_1x_3 + x_2x_3 = \dfrac{b}{k} \\[2ex]
(c) & x_1x_2x_3 = \dfrac{1-b^2}{2ak} \\[2ex]
 & \text{Multiplying (a) by (b) we get,} \\[1ex]
(d) & x_1x_2^2 + x_1x_3^2 + x_2x_1^2 + x_2x_3^2 + x_3x_1^2 + x_3x_2^2 = (x_1 + x_2 + x_3) \\
 & (x_1x_2 + x_1x_3 + x_2x_3) - 3(x_1x_2x_3) = \left(-\dfrac{a^2+k^2+2bk+1}{2ak}\right)\left(\dfrac{b}{k}\right) - \\
 & (3)\left(\dfrac{1-b^2}{2ak}\right) = \dfrac{-a^2b-bk^2+b^2k-b-3k}{2ak^2} \\[2ex]
(e) & \text{from (a)} \quad a^2+k^2+2bk+1 = -2ak(x_1 + x_2 + x_3)
\end{array}\right.$$

Substituting these values in (8) we obtain, as the numerator of U_α,

$$9a^2k^2\left(\frac{1-b^2}{2ak}\right)\left(\frac{b}{k}\right) + \cancel{6ak(a^2+k^2+2bk+1)(x_1x_2x_3)}(x_1 + x_2 + x_3) +$$

$$3a^2bk\left(\frac{-a^2b-bk^2+b^2k-b-3k}{2ak^2}\right) - \cancel{6ak(a^2+k^2+2bk+1)(x_1x_2x_3)}$$

$$\cancel{(x_1 + x_2 + x_3)} + a^2b^2\left(-\frac{a^2+k^2+2bk+1}{2ak}\right) + 2ab(a^2+k^2+2bk+1)\left(\frac{b}{k}\right)$$

$$= \frac{9ab-9ab^3}{2} + \frac{-3a^3b^2-3ab^2k^2+3ab^3k-3ab^2-9abk}{2k} +$$

$$\frac{-a^3b^2-ab^2k^2-2ab^3k-ab^2}{2k} + \frac{2a^3b^2+2ab^2k^2+4ab^3k+2ab^2}{k} =$$

$$= \frac{9abk-9ab^3k-3a^3b^2-3ab^2k^2+3ab^3k-3ab^2-9abk-a^3b^2-ab^2k^2-2ab^3k}{2k}$$

$$\frac{-ab^2+4a^3b^2+4ab^2k^2+8ab^3k+4ab^2}{2k} = \frac{0}{2k} = 0$$

HENCE $U_a = 0$

Similarly, for U_b, we have,

(10) $$U_b = y_1 x_{1b} + y_2 x_{2b} + y_3 x_{3b}$$

and since $\phi'_x \cdot x_b + \phi_b = \phi'_x \cdot x_b + (2kx^2+2ax+2b) = 0$

Therefore $$x_b = -\frac{2kx^2+2ax+2b}{6akx^2+(a^2+k^2+2bk+1)2x+2ab} =$$

$$-\frac{kx^2+ax+b}{3akx^2+(a^2+k^2+2bk+1)x+ab} = \frac{-y}{3akx^2+(a^2+k^2+2bk+1)x+ab}$$

Substituting in equation (6), we get,

(11)
$$U_b = -\left[\frac{1}{y_1}\cdot\frac{y_1}{3akx_1^2+(a^2+k^2+2bk+1)x_1+ab} + \frac{1}{y_2}\cdot\frac{y_2}{3akx_2^2+(a^2+k^2+2bk+1)x_2+ab}\right.$$
$$\left. + \frac{1}{y_3}\cdot\frac{y_3}{3akx_3^2+(a^2+k^2+2bk+1)x_3+ab}\right]$$

By elementary reductions, we obtain for the numerator of U_b,

$$9a^2k^2x_2^2x_3^2 + 3ak(a^2+k^2+2bk+1)(x_2^2x_3 + x_3^2x_2) +$$
$$3a^2bk(x_2^2+x_3^2) + (a^2+k^2+2bk+1)^2x_2x_3 + ab(a^2+k^2+2bk+1)(x_2+x_3) +$$
$$a^2b^2 + 9a^2k^2x_1^2x_3^2 + 3ak(a^2+k^2+2bk+1)(x_1^2x_3 + x_3^2x_1) + 3a^2bk(x_1^2+x_3^2)$$
$$+(a^2+k^2+2bk+1)^2x_1x_3 + ab(a^2+k^2+2bk+1)(x_1 + X_3) + a^2b^2 +$$
$$9a^2k^2x_1^2x_2^2 + 3ak(a^2+k^2+2bk+1)(x_1^2x_2 + x_2^2x_1) + 3a^2bk(x_1^2+x_2^2) +$$
$$(a^2+k^2+2bk+1)^2x_1x_2 + ab(a^2+k^2+2bk+1)(x_1+x_2) + a^2b^2 =$$

$$9a^2k^2(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2) + 3ak(a^2+k^2+2bk+1)(x_1^2x_2 + x_2^2x_1 +$$

$x_1^2x_3 + x_3^2x_1 + x_2^2x_3 + x_3^2x_2\,) + 3a^2bk(x_1^2 + x_2^2 + x_1^2 + x_3^2 + x_2^2 + x_3^2) +$
$(a^2+k^2+2bk+1)^2(x_1x_2 + x_1x_3 + x_2x_3\,) + ab(a^2+k^2+2bk+1)(x_1 + x_2 +$
$x_1 + x_3 + x_2 + x_3\,) + 3a^2b^2 \quad =$

(12) $9a^2k^2(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2) + 3ak(a^2+k^2+2bk+1)(x_1^2x_2 + x_2^2x_1$
$+x_1^2x_3 + x_3^2x_1 + x_2^2x_3 + x_3^2x_2\,) + 6a^2bk(x_1^2 + x_2^2 + x_3^2) + (a^2+k^2+2bk+1)^2$
$(x_1x_2 + x_1x_3 + x_2x_3\,) + 2ab(a^2+k^2+2bk+1)(x_1 + x_2 + x_3\,) + 3a^2b^2$

From (3) we have,

$$(13)\begin{cases} x_1 + x_2 + x_3 = -\dfrac{a^2+k^2+2bk+1}{2ak} \\ x_1x_2 + x_1x_3 + x_2x_3 = \dfrac{b}{k} \\ x_1x_2x_3 = \dfrac{1-b^2}{2ak} \\ x_1^2 + x_2^2 + x_3^2 = (x_1 + x_2 + x_3\,)^2 - 2(x_1x_2 + x_1x_3 + x_2x_3\,) \\ x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2 = (x_1x_2 + x_1x_3 + x_2x_3\,)^2 - 2(x_1x_2x_3\,)(x_1 + x_2 + x_3\,) \\ x_1^2x_2 + x_2^2x_1 + x_1^2x_3 + x_3^2x_1 + x_2^2x_3 + x_3^2x_2 = (x_1 + x_2 + x_3\,)(x_1x_2 + x_1x_3 + x_2x_3\,) - 3(x_1x_2x_3\,) \end{cases}$$

Substituting these values in (12) we obtain, as numerator of U_b,

$$9a^2k^2\left[(x_1x_2 + x_1x_3 + x_2x_3\,)^2 - 2(x_1x_2x_3\,)(x_1 + x_2 + x_3\,)\right] + 3ak(a^2+k^2+2bk+1)\left[(x_1 + x_2 + x_3\,)(x_1x_2 + x_1x_3 + x_2x_3\,) - 3(x_1x_2x_3\,)\right] + 6a^2bk\left[(x_1 + x_2 + x_3\,)^2 - 2(x_1x_2 + x_1x_3 + x_2x_3\,)\right] + (a^2+k^2+2bk+1)^2(x_1x_2 + x_1x_3 + x_2x_3\,) + 2ab(a^2+k^2+2bk+1)(x_1 + x_2 + x_3\,) + 3a^2b^2 \quad =$$

$$9a^2k^2\left[\frac{b^2}{k^2} - 2\left(\frac{1-b^2}{2ak}\right)\left(-\frac{a^2+k^2+2bk+1}{2ak}\right)\right] +$$

$$3ak(a^2+k^2+2bk+1)\left[\left(-\frac{a^2+k^2+2bk+1}{2ak}\right)\left(\frac{b}{k}\right)-3\left(\frac{1-b^2}{2ak}\right)\right]+$$

$$6a^2bk\left[\left(-\frac{a^2+k^2+2bk+1}{2ak}\right)^2-\frac{2b}{k}\right]+\cancel{(a^2+k^2+2bk+1)^2\left(\frac{b}{k}\right)}+$$

$$\cancel{2ab(a^2+k^2+2bk+1)\left(-\frac{a^2+k^2+2bk+1}{2ak}\right)}+3a^2b^2$$

The quantities crossed in red are equal and opposite in sign and vanish. Partially multiplying the remaining terms out, we get, as the numerator of U_b,

$$9a^2b^2+\frac{9(1-b^2)(a^2+k^2+2bk+1)}{2}-\frac{3b(a^2+k^2+2bk+1)^2}{2k}-$$

$$\frac{9(1-b^2)(a^2+k^2+2bk+1)}{2}+\frac{3b(a^2+k^2+2bk+1)^2}{2k}-12a^2b^2+3a^2b^2=0$$

Therefore $U_b = 0$

HENCE $U_a = 0$, $U_b = 0$ and we have therefore,

$U(ab)$ = a constant = C

We have, then,

$$(14)\quad \int_0^{x_1}\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}+\int_0^{x_2}\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}+\int_{\frac{1}{k}}^{x_3}\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}=C$$

Determination of C

If, in (3), we put $b = 1$ and $a = 0$, it reduces to

$(k+1)^2x^2 = 0$ <u>or</u> $x^2 = 0$, i.e. two roots of the cubic are 0 and the third increases without bound, giving $x_1 = x_2 = 0, x_3 = \infty$

Substituting in (14) we have,

$$(15)\quad \int_0^0 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} + \int_0^0 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} + \int_{\frac{1}{k}}^{\infty} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = C$$

The first two integrals in (15) vanish identically, and putting $x = \frac{1}{kt}$, $dx = -\frac{dt}{kt^2}$, the third integral becomes,

$$-\int_1^{t_3} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = C, \quad \text{As } x \to \infty,\ t_3 = \frac{1}{kx_3} \to \frac{1}{\infty} = 0$$

Consequently, $\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = C$, and (14) becomes,

$$(16)\quad \int_0^{x_1} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} + \int_0^{x_2} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} - \int_1^{t_3} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} =$$

$$\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}.$$

But, $\int_1^{t_3} f(t)dt = \int_1^0 f(t)dt + \int_0^{t_3} f(t)dt = -\int_0^1 f(t)dt + \int_0^{t_3} f(t)dt$

Therefore substitution in (16) gives,

$$\int_0^{x_1} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} + \int_0^{x_2} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \int_0^{t_3} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

Replacing t_3 by $\frac{1}{kx_3}$, we get,

$$(17)\quad \int_0^{x_1} \frac{dx}{\sqrt{(1-x^2)(1-k^2X^2)}} + \int_0^{x_2} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \int_0^{\frac{1}{kx_3}} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

From equation (3), $x_1 x_2 x_3 = \dfrac{1-b^2}{2ak}$

Therefore $kx_1 x_2 = \dfrac{1-b^2}{2ax_3}$, and $k^2x_1^2x_2^2 = \dfrac{(1-b^2)^2}{4a^2x_3^2}$

By Composition, $1 - k^2x_1^2x_2^2 = 1 - \dfrac{(1-b^2)^2}{4a^2x_3^2} = \dfrac{4a^2x_3^2-(1-b^2)^2}{4a^2x_3^2}$

Or, $4a^2x_3^2 = \dfrac{4a^2x_3^2 - (1-b^2)^2}{1-k^2x_1^2x_2^2}$, Therefore, we have,

$$(18) \quad x_3 = \frac{\dfrac{4a^2x_3^2 - (1-b^2)^2}{4a^2x_3}}{1-k^2x_1^2x_2^2} = \frac{\dfrac{4a^2kx_3^2 - k(1-b^2)^2}{4a^2kx_3}}{1-k^2x_1^2x_2^2}$$

By multiplying both sides of (18) by $\dfrac{1}{kx_3^2}$, we obtain,

$$(19) \quad \frac{1}{kx_3} = \frac{\dfrac{4a^2kx_3^2 - k(1-b^2)^2}{4a^2k^2x_3^3}}{1-k^2x_1^2x_2^2}$$

By separating the numerator of (19) in two parts, and adding and subtracting $\dfrac{b^2}{kx_3}$, we get,

$$(20) \quad \frac{1}{kx_3} = \frac{\dfrac{1}{kx_3} - \dfrac{k(1-b^2)^2}{4a^2k^2x_3^3} + \dfrac{b^2}{kx_3} - \dfrac{b^2}{kx_3}}{1 - k^2x_1^2x_2^2}$$

If, to the numerator of (20), we add and subtract $\dfrac{b(1-b^2)}{2akx_3^2}$, it becomes,

$$(21) \quad \frac{1}{kx_3} = \frac{\dfrac{b(1-b^2)}{2akx_3^2} - \dfrac{k(1-b^2)^2}{4a^2k^2x_3^3} + \dfrac{1}{kx_3} - \dfrac{b^2}{kx_3} + \dfrac{b^2}{kx_3} - \dfrac{b(1-b^2)}{2akx_3^2}}{1 - k^2x_1^2x_2^2}$$

Combining the third and fourth terms of the numerator of

(21) and multiplying and dividing the result by $2a$, we get,

$$(22)\quad \frac{1}{kx_3} = \frac{\dfrac{b(1-b^2)}{2akx_3^2} - \dfrac{k(1-b^2)^2}{4a^2k^2x_3^3} + \dfrac{2a(1-b^2)}{2akx_3} + \dfrac{b^2}{kx_3} - \dfrac{b(1-b^2)}{2akx_3^2}}{1 - k^2x_1^2x_2^2}$$

which may be written,

$$(23)\quad \frac{1}{kx_3} = \frac{\dfrac{b\left(\dfrac{1-b^2}{2akx_3}\right) - k\left(\dfrac{(1-b^2)^2}{4a^2k^2x_3^2}\right)}{x_3} + 2a\left(\dfrac{1-b^2}{2akx_3}\right) + \dfrac{b\left(\dfrac{b}{k} - \dfrac{1-b^2}{2akx_3}\right)}{x_3}}{1 - k^2x_1^2x_2^2}$$

From $x_1x_2x_3 = \dfrac{1-b^2}{2ak}$ we get,

$$(24)\quad \frac{1-b^2}{2akx_3} = x_1x_2 \quad \text{and} \quad (25)\quad \frac{(1-b^2)^2}{4a^2k^2x_3^2} = x_1^2x_2^2$$

Substituting these values in (23) we obtain,

$$(26)\quad \frac{1}{kx_3} = \frac{\dfrac{bx_1x_2 - kx_1^2x_2^2}{x_3} + 2ax_1x_2 + \dfrac{b\left(\dfrac{b}{k} - x_1x_2\right)}{x_3}}{1 - k^2x_1^2x_2^2}$$

and, changing form to obtain the quotient $\dfrac{\frac{b}{k} - x_1x_2}{x_3}$,

(26) becomes,

$$(27)\quad \frac{1}{kx_3} = \frac{kx_1x_2\left(\dfrac{\frac{b}{k} - x_1x_2}{x_3}\right) + 2ax_1x_2 + b\left(\dfrac{\frac{b}{k} - x_1x_2}{x_3}\right)}{1 - k^2x_1^2x_2^2}$$

From (3), $x_1x_2 + x_1x_3 + x_2x_3 = \dfrac{b}{k}$

$$x_1 x_2 + x_3(x_1 + x_2) = \frac{b}{k}, \quad \text{therefore} \quad x_3(x_1 + x_2) = \frac{b}{k} - x_1 x_2$$

(28) Hence, $$\frac{\frac{b}{k} - x_1 x_2}{x_3} = x_1 + x_2$$

Using (28) , (27) becomes,

$$\frac{1}{kx_3} = \frac{kx_1 x_2(x_1 + x_2) + 2ax_1 x_2 + b(x_1 + x_2)}{1 - k^2 x_1^2 x_2^2}$$

$$= \frac{kx_1^2 x_2 + kx_1 x_2^2 + ax_1 x_2 + ax_1 x_2 + bx_1 + bx_2}{1 - k^2 x_1^2 x_2^2}$$

$$= \frac{(kx_1 x_2^2 + ax_1 x_2 + bx_1) + (kx_1^2 x_2 + ax_1 x_2 + bx_2)}{1 - k^2 x_1^2 x_2^2}$$

(29) $$\frac{1}{kx_3} = \frac{x_1(kx_2^2 + ax_2 + b) + x_2(kx_1^2 + ax_1 + b)}{1 - k^2 x_1^2 x_2^2}$$

But, since x_1 and x_2 are roots of,

$$kx^2 + ax + b - \sqrt{(1-x^2)(1-k^2x^2)} = 0,$$

$$kx_1^2 + ax_1 + b = \sqrt{(1-x_1^2)(1-k^2x_1^2)}$$

and, $$kx_2^2 + ax_2 + b = \sqrt{(1-x_2^2)(1-k^2x_2^2)}$$

therefore, substituting in (29), we get,

(30) $$\frac{1}{kx_3} = x_3' = \frac{x_1\sqrt{(1-x_2^2)(1-k^2x_2^2)} + x_2\sqrt{(1-x_1^2)(1-k^2x_1^2)}}{1 - k^2 x_1^2 x_2^2}$$

Consequently, (17) becomes,

$$\int_0^{x_1} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} + \int_0^{x_2} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \int_0^{x_3'} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

or, $F(k, x_1) + F(k, x_2) = F(k, x_3')$ where x_3' has the value given in (30).

This is, "THE ADDITION THEOREM FOR THE LEGENDRE ELLIPTIC INTEGRAL OF THE FIRST KIND .", $F(k,x)$.

For the special values of the cubic, $a = 0$ and $b = 1$, the variable curve takes the position on the graph as shown below: